Know Math to teach the Common Core Math-

the story of progressions

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What are Progressions

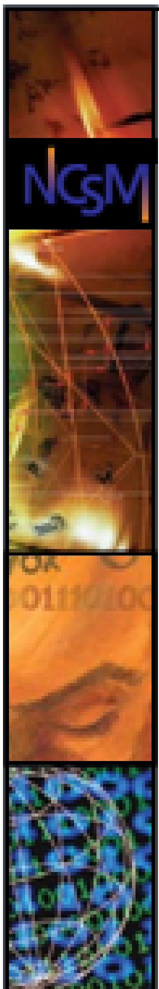
The Common Core State Standards in mathematics were built on progressions: narrative documents describing the progression of a topic ac a number of grade levels, informed both by research on children's cognitiv development and by the logical structure of mathematics.

They can explain why standards are sequenced the way they are, point out cognitive difficulties and pedagogical solutions, and give more detail on particularly knotty areas of the mathematics. This would be useful in teache preparation and professional development, organizing curriculum, and writing textbooks.

- Domain Specific
- Demonstrate coherency
- Essence of Common Core
- Address the Standards of Mathematical Practice
- http://ime.math.arizona.edu/progressions/

Progression: 3-5 Number and Operations–Fractions

- observe the models
- connect the models to the content standards
- identify the SMP at work
- understand why "knowing math" is the answer to teaching CC



Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 3

Grade 3 students start with unit fractions (fractions with numerator 1), which are formed by partitioning a whole into equal parts and taking one part, e.g., if a whole is partitioned into 4 equal parts then each part is $\frac{1}{4}$ of the whole, and 4 copies of that part make the whole. Next, students build fractions from unit fractions, seeing the numerator 3 of $\frac{3}{4}$ as saying that $\frac{3}{4}$ is the quantity you get by putting 3 of the $\frac{1}{4}$'s together.[3.NF.1] They read any fraction this way, and in particular there is no need to introduce "proper fractions" and "improper fractions" initially; $\frac{5}{3}$ is the quantity you get by combining 5 parts together when the whole is divided into 3 equal parts.

Two important aspects of fractions provide opportunities for the mathematical practice of attending to precision (MP6):

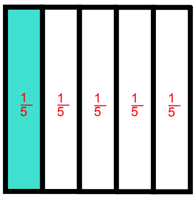
- Specifying the whole.
- Explaining what is meant by "equal parts."

2.G.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

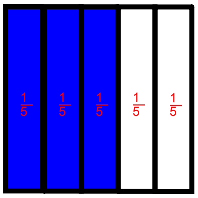
(Coherence, SMP 2, SMP 7)

3.NF.1 Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

$\frac{1}{5}$

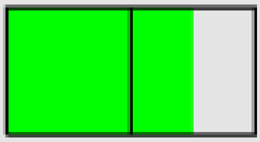


$\frac{3}{5}$



(SMP 2, 7)

The importance of specifying the whole



(SMP 6)

Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$*; if the entire rectangle is the whole, the shaded area represents* $\frac{3}{4}$.

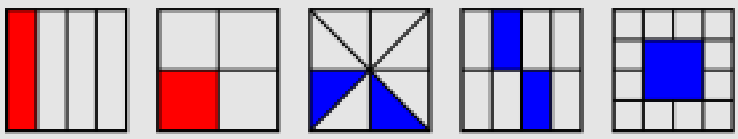
Initially, students can use an intuitive notion of congruence ("same size and same shape") to explain why the parts are equal, e.g., when they divide a square into four equal squares or four equal rectangles.

Students come to understand a more precise meaning for "equal parts" as "parts with equal measurements." For example, when a ruler is partitioned into halves or quarters of an inch, they see that each subdivision has the same length. In area models they reason about the area of a shaded region to decide what fraction of the whole it represents (MP3). (SMP 6)

The goal is for students to see unit fractions as the basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers; just as every whole number is obtained by combining a sufficient number of 1s, every fraction is obtained by combining a sufficient number of unit fractions.

(SMP 1,2,7,8)

Area representations of $\frac{1}{4}$



In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.

3.NF.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram.

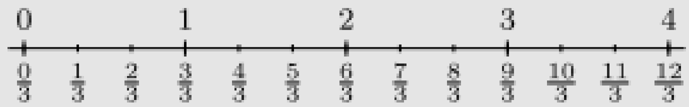
a Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

b Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

The number line

(SMP 4 and SMP 5)

The number line marked off in thirds



3.NF.3abc Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

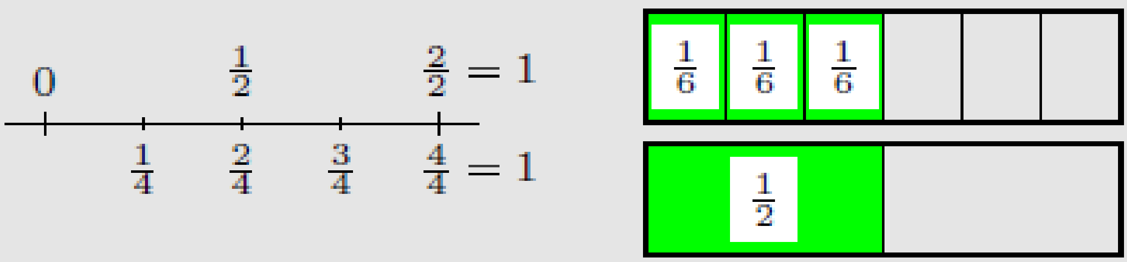
a Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4$, $4/6 = 2/3$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

(SMP 7)

Using the number line and fraction strips to see fraction equivalence

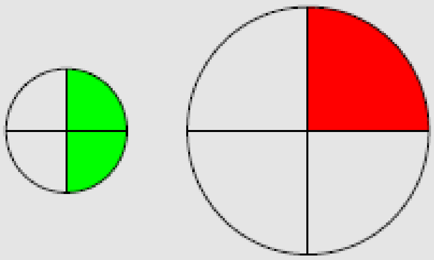


2.MD.3 Estimate lengths using units of inches, feet, centimeters, and meters. (coherence)

3.NF.3d Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

d Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model. (SMP 3)

The importance of referring to the same whole when comparing fractions



(SMP 6)

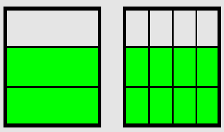
A student might think that $\frac{1}{4} > \frac{1}{2}$, *because a fourth of the pizza on the right is bigger than a half of the pizza on the left.*

Grade 4

Grade 4 students learn a fundamental property of equivalent fractions: multiplying the numerator and denominator of a fraction by the same non-zero whole number results in a fraction that represents the same number as the original fraction. This property forms the basis for much of their other work in Grade 4, including the comparison, addition, and subtraction of fractions and the introduction of finite decimals.

4.NF.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

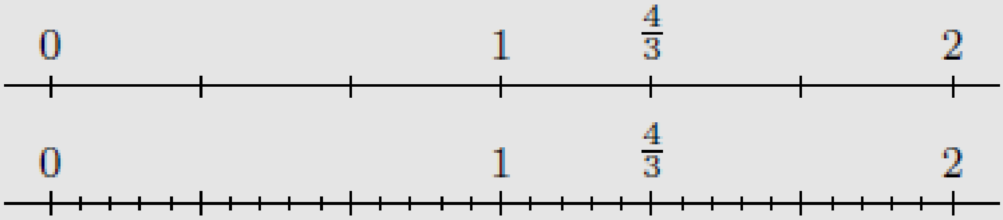
Using an area model to show that $\frac{2}{3} = \frac{4 \times 2}{4 \times 3}$



(SMP 1,2,3,4,7)

The whole is the square, measured by area. On the left it is divided horizontally into 3 rectangles of equal area, and the shaded region is 2 of these and so represents $\frac{2}{3}$. On the right it is divided into 4×3 small rectangles of equal area, and the shaded area comprises 4×2 of these, and so it represents $\frac{4 \times 2}{4 \times 3}$.

Using the number line to show that $\frac{4}{3} = \frac{5\times4}{5\times3}$



$\frac{4}{3}$ is 4 parts when each part is $\frac{1}{3}$, and we want to see that this is also 5×4 parts when each part is $\frac{1}{5\times3}$. Divide each of the intervals of length $\frac{1}{3}$ into 5 parts of equal length. There are 5×3 parts of equal length in the unit interval, and $\frac{4}{3}$ is 5×4 of these. Therefore $\frac{4}{3} = \frac{5\times4}{5\times3} = \frac{20}{15}$.

(SMP 2,8)

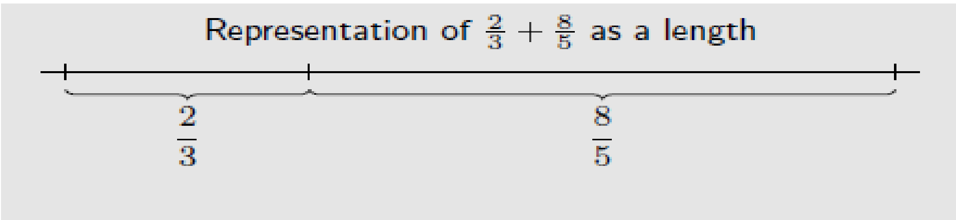
This concept is repeated in 5th grade in an area model

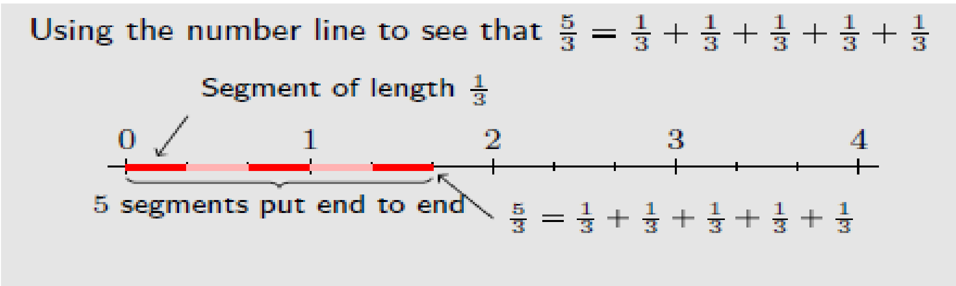
4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

a Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

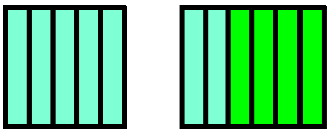
c Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.





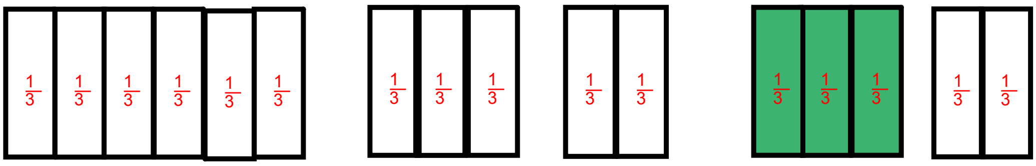
Which SMP's do you see here?

Using Models to gain understanding of various kinds of fractions



$$\frac{7}{5}+\frac{4}{5} = \overbrace{\frac{1}{5}+\cdots\frac{1}{5}}^{7}+\overbrace{\frac{1}{5}+\cdots\frac{1}{5}}^{4} = \frac{\overbrace{1+1+\cdots+1}^{7+4}}{5} = \frac{7+4}{5}.$$

(SMP 2, 3)



$$\frac{5}{3}=\frac{3}{3}+\frac{2}{3}=1+\frac{2}{3}=1\frac{2}{3}.$$

How would you show? $7\frac{1}{5}=7+\frac{1}{5}=\frac{35}{5}+\frac{1}{5}=\frac{36}{5}.$

It is not necessary to find a least common denominator to calculate sums of fractions, and in fact the effort of finding a least common denominator is a distraction from understanding algorithms for adding fractions.

4.NF.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[1]

Decimals Fractions with denominator 10 and 100, called *decimal fractions*, arise naturally when student convert between dollars and cents, and have a more fundamental importance, developed in Grade 5, in the base 10 system. For example, because there are 10 dimes in a dollar, 3 dimes is $\frac{3}{10}$ of a dollar; and it is also $\frac{30}{100}$ of a dollar because it is 30 cents, and there are 100 cents in a dollar. Such reasoning provides a concrete context for the fraction equivalence

$$\frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100}.$$

(SMP 2, 7, 8)

Grade 3 students learn to add decimal fractions by converting them to fractions with the same denominator, in preparation for general fraction addition in Grade 5:[4.NF.5]

$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}.$$

(coherence)

They can interpret this as saying that 3 dimes together with 27 cents make 57 cents.

Fractions with denominators equal to 10, 100, etc., such

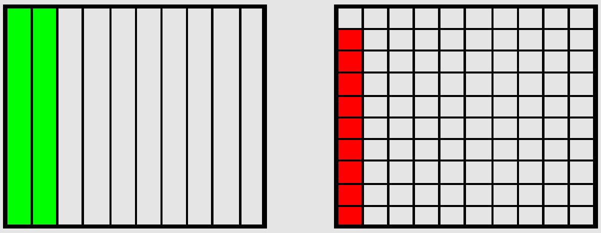
$$\frac{27}{10}, \quad \frac{27}{100}, \quad \text{etc.}$$

can be written by using a *decimal point* as[4.NF.6]

$$2.7, \quad 0.27.$$

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual model.

Seeing that $0.2 > 0.09$ using a visual fraction model



The shaded region on the left shows 0.2 *of the unit square, since it is two parts when the square is divided int 10 parts of equal area. The shaded region on the right shows* 0.09 *of the unit square, since it is 9 parts when the unit is divided into 100 parts of equal area.*

(SMP 1, 3, 4)

Grade 5

10

Grade 5

Adding and subtracting fractions In Grade 4, students calculate sums of fractions with different denominators where one denominator is a divisor of the other, so that only one fraction has to be changed. For example, they might have used a fraction strip to reason that

$$\frac{1}{3}+\frac{1}{6}=\frac{2}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2},$$

and in working with decimals they added fractions with denominators 10 and 100, such as

$$\frac{2}{10}+\frac{7}{100}=\frac{20}{100}+\frac{7}{100}=\frac{27}{100}.$$

They understand the process as expressing both summands in terms of the same unit fraction so that they can be added. Grade 5 students extend this reasoning to situations where it is necessary to re-express both fractions in terms of a new denominator.[5.NF.1] For example, in calculating $\frac{2}{3}+\frac{5}{4}$ they reason that if each third in $\frac{2}{3}$ is subdivided into fourths, and if each fourth in $\frac{5}{4}$ is subdivided into thirds, then each fraction will be a sum of unit fractions with denominator $3\times 4=4\times 3=12$:

$$\frac{2}{3}+\frac{5}{4}=\frac{2\times 4}{3\times 4}+\frac{5\times 3}{4\times 3}=\frac{8}{12}+\frac{15}{12}=\frac{23}{12}.$$

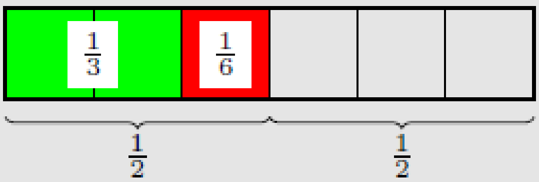
It is not necessary to find a least common denominator to calculate sums of fractions, and in fact the effort of finding a least common denominator is a distraction from understanding algorithms for adding fractions.

(SMP 1)

Students make sense of fractional quantities when solving word problems, estimating answers mentally to see if they make sense.[5.NF.2] For example in the problem

5.NF.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

Using a fraction strip to show that $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$



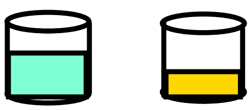
Students discover patterns and then begin to generalize (SMP 7,8)

5.NF.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

For example in the problem

> Ludmilla and Lazarus each have a lemon. They need a cup of lemon juice to make hummus for a party. Ludmilla squeezes $\frac{1}{2}$ a cup from hers and Lazarus squeezes $\frac{2}{5}$ of a cup from his. How much lemon juice to they have? Is it enough?

students estimate that there is almost but not quite one cup of lemon juice, because $\frac{2}{5} < \frac{1}{2}$. They calculate $\frac{1}{2} + \frac{2}{5} = \frac{9}{10}$, and see this as $\frac{1}{10}$ less than 1, which is probably a small enough shortfall that it will not ruin the recipe. They detect an incorrect result such as $\frac{2}{5} + \frac{2}{5} = \frac{3}{7}$ by noticing that $\frac{3}{7} < \frac{1}{2}$.



(SMP 1, 2, 3, 4, 5, 6)

Multiplying and dividing fractions In Grade 4 students connected fractions with addition and multiplication, understanding that

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 5 \times \frac{1}{3}.$$

(coherence)

In Grade 5, they connect fractions with division, understanding that

$$5 \div 3 = \frac{5}{3},$$

or, more generally, $\frac{a}{b} = a \div b$ for whole numbers a and b, with b not equal to zero.[5.NF.3] They can explain this by working with their understanding of division as equal sharing (see figure in margin). They also create story contexts to represent problems involving division of whole numbers. For example, they see that

> If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?

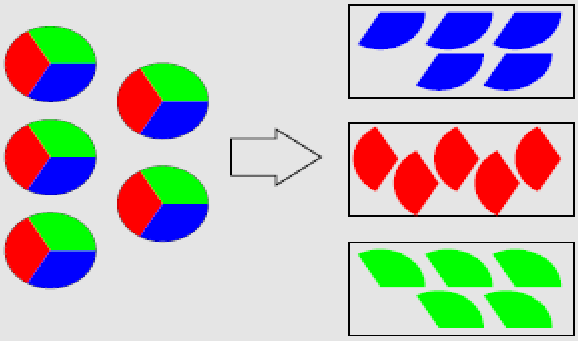
One option: Share 50 pounds among 9 people (50 x 1/9= 50/9

Another way: 9 x 5= 45 pounds, each person gets 5 pounds and the extra 5 pounds is divided between them 5/9, so each person gets 5 $^{5/9}$ pounds.

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

How to share 5 objects equally among 3 shares:

$$5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$$

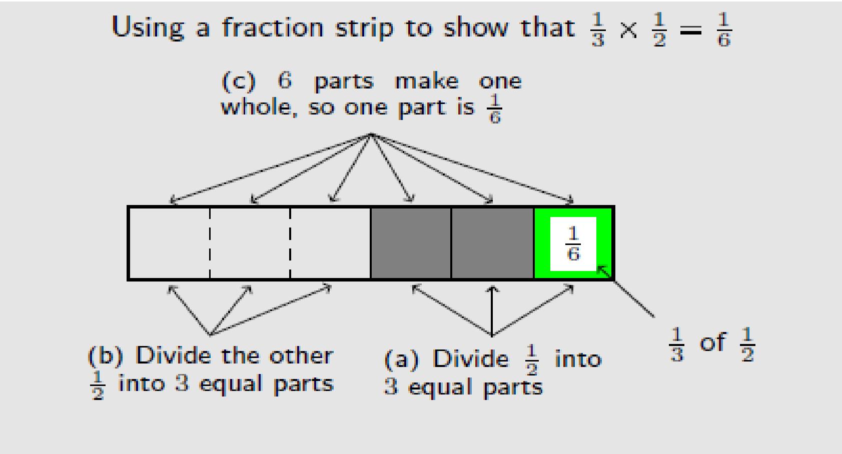


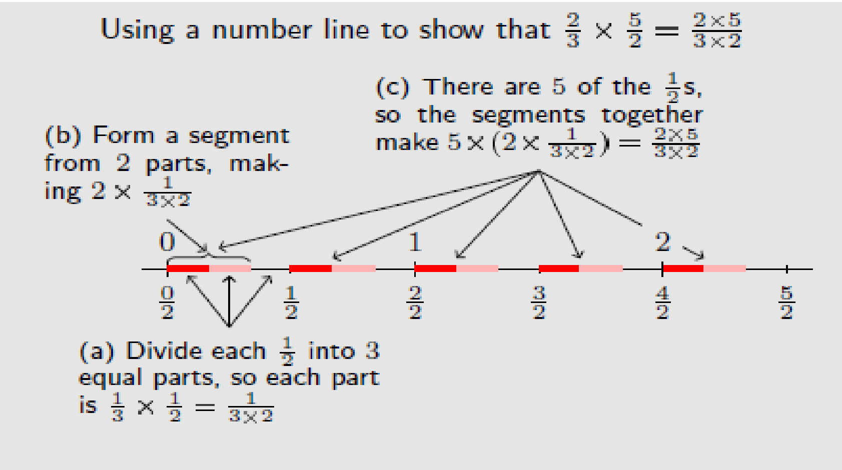
If you divide 5 objects equally among 3 shares, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object, and so each share is $5 \times \frac{1}{3} = \frac{5}{3}$ of an object.

(SMP 3)

5.NF.4a Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.



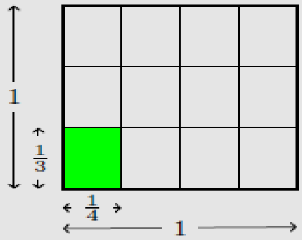


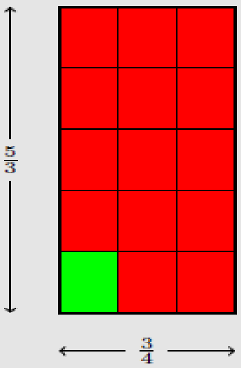
5.NF.4b Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Using an area model to show that $\frac{3}{4} \times \frac{5}{3} = \frac{3 \times 5}{4 \times 3}$

Because 4×3 rectangles $\frac{1}{4}$ wide and $\frac{1}{3}$ high fit in a unit square, $\frac{1}{4} \times \frac{1}{3} = \frac{1}{4 \times 3}$.





The rectangle of width $\frac{3}{4}$ and height $\frac{5}{3}$ is tiled with 3×5 rectangles of area $\frac{1}{4 \times 3}$, so has area $\frac{3 \times 5}{4 \times 3}$.